Math 3B — Week 1

Integrals by Definition For each of the following, write the integral as a Riemann sum or determine what integral the Riemann sum is describing.

(a) $\displaystyle\int_0^\pi x^2\tan(x)\,dx$

(b) $\displaystyle\int_1^2 x^2+x^3\,dx$

(c) $\displaystyle\int_2^4 \left(x+x^2\right)^2\,dx$

(d) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^n \frac{2}{n}\left(3-\left(5+\frac{2i}{n}\right)^2\right)$

(e) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^n \frac{e^{i/n}}{n+i}$

(f) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^n -\frac{5\sin\left(-\frac{5i}{n}\right)}{n}$

(a) Since $a=0$ and $b=\pi$, we can say

$$\Delta x=\frac{b-a}{n}=\frac{\pi}{n}.$$

Since $f(x)=x^2\tan(x)$, we have

$$f(x_i)=f(a+i\Delta x)=f\left(0+\frac{i\pi}{n}\right)=\left(\frac{i\pi}{n}\right)^2\tan\left(\frac{i\pi}{n}\right).$$

This gives us a Riemann sum

$$\begin{aligned}\int_0^\pi x^2\tan(x)\,dx&=\lim_{n\to\infty}\sum_{i=1}^n\frac{\pi}{n}\left(\frac{i\pi}{n}\right)^2\tan\left(\frac{i\pi}{n}\right)\\&=\lim_{n\to\infty}\sum_{i=1}^n\frac{i^2\pi^3\tan\left(\frac{i\pi}{n}\right)}{n^3}.\end{aligned}$$

Note the simplification at the end is not necessary. However, it may be helpful to you to simplify Riemann sums so that when you have to go backwards, simplified expressions look more familiar!

(b) Once again, we have $a=1$, $b=2$, and $f(x)=x^2+x^3$, so we find

$$\begin{aligned}\Delta x&=\frac{1}{n},\\ f(x_i)&=\left(1+\frac{i}{n}\right)^2+\left(1+\frac{i}{n}\right)^3,\end{aligned}$$

so

$$\int_1^2 x^2+x^3\,dx=\lim_{n\to\infty}\sum_{i=1}^n\frac{1}{n}\left[\left(1+\frac{i}{n}\right)^2+\left(1+\frac{i}{n}\right)^3\right].$$

(c) And again, we have $a = 2$, $b = 4$, $f(x) = (x + x^2)^2$, so

$$\Delta x = \frac{2}{n},$$
$$f(x_i) = \left(2 + \frac{2i}{n} + \left(2 + \frac{2i}{n}\right)^2\right)^2,$$

and

$$\int_2^4 \left(x + x^2\right)^2 \, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(2 + \frac{2i}{n} + \left(2 + \frac{2i}{n}\right)^2\right)^2.$$

(d) Going in this direction is less straightforward than in the previous three questions. Luckily, the expression in this part is left in a way that makes it relatively easy to read off the various parts. We know we want to write the given Riemann sum as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i),$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$. A nice candidate to take the place of Δx for us is $2/n$ (why?). If we take $\Delta x = 2/n$, then we have

$$x_i = a + i\Delta x = a + i\frac{2}{n}.$$

Looking back at the given expression, we notice that the $5 + 2i/n$ already looks like this, so let's take that as our x_i. Thus far, we have found

$$\Delta x = \frac{2}{n} = \frac{b - a}{n},$$
$$x_i = 5 + \frac{2i}{n} = a + i\Delta x.$$

From Δx, we find that $b - a = 2$. From x_i, we find $a = 5$. Putting these together, we find that $b = 7$. We now have the bounds of our integral! To find our function $f(x)$, we need only look at everything that is left in our Riemann sum. Since the summands are supposed to look like $\Delta x \cdot f(x_i)$ and we already said $\Delta x = 2/n$, we are left with

$$f(x_i) = 3 - \left(5 + \frac{2i}{n}\right)^2.$$

But we know that $5 + 2i/n$ is our x_i, so we conclude that

$$f(x) = 3 - x^2$$

(verify that using this f gives us the correct $f(x_i)$). We have now found all the necessary parts (a, b, and f), so we can say

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(3 - \left(5 + \frac{2i}{n}\right)^2\right) = \int_5^7 3 - x^2 \, dx.$$

(e) This problem is trickier than the previous one. We're going to end up having to rewrite the given expression into a form that looks more like what we want before figuring out what integral it represents. The first thing that might strike us about this integral is the $n + i$ in the denominator. This is a little strange since when we take $f(x_i)$, we are expecting i/n sorts of terms. This feeling becomes stronger when we see the i/n in the numerator. Let us suppose that $f(x)$ has something to do with e^x. When we take $f(x_i)$, we expect to find something that looks like $e^{a+i\Delta x}$. In our case, we conjecture that

$$x_i = \frac{i}{n}$$

so that $a = 0$ and $\Delta x = 1/n$. Since $a = 0$ and $b - a = 1$, we conclude that $b = 1$. Now all that's left is to find $f(x)$. To do so, recall that we want the Riemann sum to be written as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i).$$

Well, we know $\Delta x = 1/n$, so let us factor out an n from the denominator so we can write a $1/n$ in front of the rest of the expression:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{e^{i/n}}{1 + \frac{i}{n}}.$$

Now since $\Delta x = 1/n$, we must have

$$f(x_i) = \frac{e^{i/n}}{1 + \frac{1}{n}}.$$

This is nice because we already said that $x_i = i/n$, so we just have

$$f(x) = \frac{e^x}{1 + x}.$$

Finally, we may say

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{i/n}}{n + i} = \int_0^1 \frac{e^x}{1 + x}\, dx.$$

(f) This last problem is actually not as bad as the previous problem. It is not too hard to see that

$$\lim_{n\to\infty} \sum_{i=1}^{n} -\frac{5\sin\left(-\frac{5i}{n}\right)}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} -\frac{5}{n} \sin\left(-\frac{5i}{n}\right).$$

Now, we can just take

$$\Delta x = -\frac{5}{n},$$
$$x_i = -\frac{5i}{n}$$

to find $a = 0$ and $b - a = -5$, so $b = -5$. Our function is the only thing that's left, namely $\sin(x)$. This leaves us with

$$\lim_{n\to\infty} \sum_{i=1}^{n} -\frac{5\sin\left(-\frac{5i}{n}\right)}{n} = \int_0^{-5} \sin(x)\, dx.$$

This is weird...the upper integration bound is smaller than the lower...more on that in the next part!

Integrals as Areas Evaluate each of the following integrals geometrically.

(a) $\int_0^1 x\,dx$

(b) $\int_0^1 1\,dx$

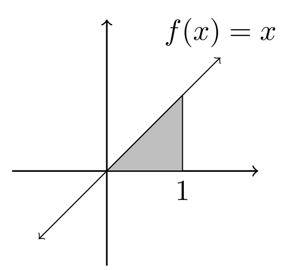
(c) $\int_0^1 -1\,dx$

(d) $\int_1^0 1\,dx$

(e) $\int_1^0 -1\,dx$

(f) $\int_0^{2\pi} \sin(x)\,dx$

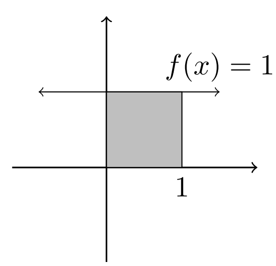
(a) We are just looking for the following area:



Since the region is a triangle with width 1 and height 1, we conclude that

$$\int_0^1 x\,dx = \frac{1}{2}(1)(1) = \frac{1}{2}.$$

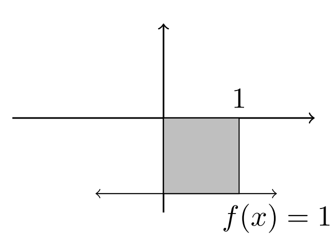
(b) This one is even easier.



Since the region is a square with width 1 and height 1, we conclude that

$$\int_0^1 1\,dx = 1(1) = 1.$$

(c) This is where the integrals begin to get interesting!



Of course, the shaded region still has an area of 1, but integrals calculate *signed* area. Since the area is below the x-axis (or equivalently since $f(x)$ is negative), we have a "negative area". This gives us

$$\int_0^1 -1\,dx = -1.$$

(d) This integral is strange. We're looking at the same region as in part (b), but this time, we're integrating backwards! Instead of our upper bound being greater than our lower bound, our upper bound is smaller than our lower bound! Integrating "backwards" also produces a "negative area", so we have

$$\int_1^0 1\,dx = -1.$$

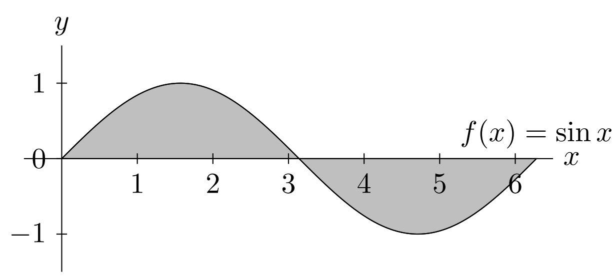
In general, it is true that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

(e) Now we're putting all the strange things together! We're integrating a negative area backwards, so the negatives cancel out and we are left with

$$\int_1^0 -1\,dx = -(-1) = 1.$$

(f) This one is not a super simple function like the previous ones, but we can still draw it and see what happens.



Now, we can't be sure without actually calculating it, but it seems like the positive area and the negative area are roughly the same! Indeed, if we actually compute the integral, we will see that

$$\int_0^{2\pi} \sin(x)\,dx = 0.$$